Deletion Code Bounds via Pseudorandomness

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February 10, 2023

Setting the Stage

The Beginning

Noisy Channel Encoding Theorem (Shannon ’48)

For a binary symmetric channel with error rate $p \in (0, 1)$, let $C = 1 - H(p)$. For any rate $R < C$, there exists a code in $\{0, 1\}^n$ of size 2^{Rn} that w.h.p. correctly transmits information.

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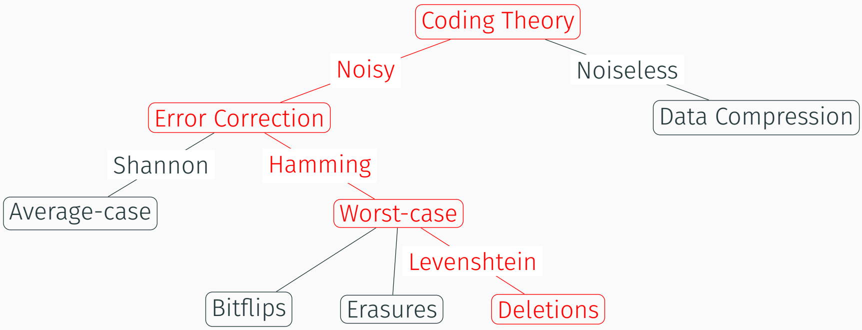
He wants to create a method of coding, but he doesn't know what to do so he makes a random code. Then he is stuck. ***And then he asks the impossible question, "What would the average random code do?"*** *He then proves that the average code is arbitrarily good, and that therefore there must be at least one good code. Who but a man of infinite courage could have dared to think those thoughts?*

— Richard Hamming, on Claude Shannon.

Phylogeny of Coding Theory



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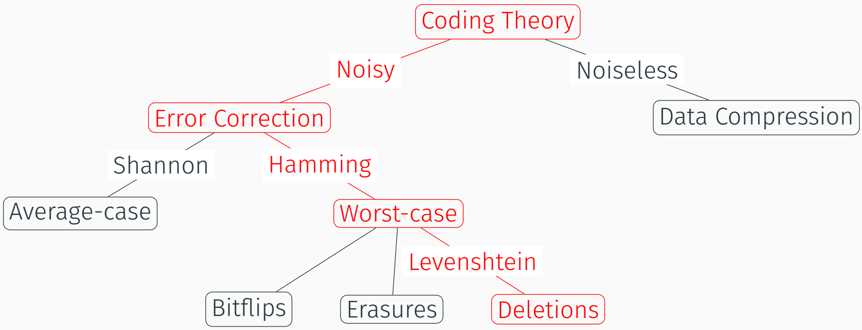

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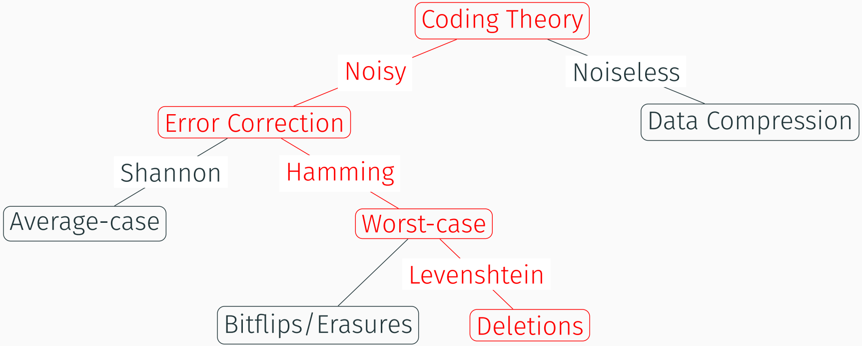

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Noise Models

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Electromagnetic signal

Bitflip errors 1101 $\rightarrow$ 1001

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Electromagnetic signal	Bitflip errors $1101 \to 1001$
Auditory experience	Erasure errors $1101 \to 1?01$
Transcribed lyrics	Deletion errors $1101 \to 101$

Background: Bitflips and Erasures

Definition

A *code of length n and distance d* is a subset $C \subseteq \{0,1\}^n$ such that $\min d_{\text{Hamming}}(s,t) = d$ over all distinct $s, t \in C$. Let $A(n,d)$ denote the size of the largest such code.

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(1) When d is fixed and $n \to \infty$, what is the order of $A(n,d)$?

(2) For which $p \in (0,1)$ is $A(n,pn) \geq 2^{\Omega(n)}$? A code with size $2^{\Omega(n)}$ is called "positive rate."

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(For $p \geq \frac{1}{2}$, $A(n, pn) \leq 2n$.)

Deletion Codes

Worst-Case Deletion Errors

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Let $\Gamma_{n,d}$ be the **confusability graph** on $\{0,1\}^n$ defined by $s \sim t$ if $\mathrm{LCS}(s,t) \geq n - d$. A deletion code is just an independent set in $\Gamma_{n,d}$.

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Theorem (Bukh, Guruswami, Håstad '16)

There exist explicit, efficient pn-deletion codes up to $p^* \geq \sqrt{2} - 1 \approx .414$.

Bounds

Constant number of errors.

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Our Results

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Linear number of errors. (Guruswami, H., Li '22)

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Theorem (Alon, Bourla, Graham, H., Kravitz '22)

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Observe that $D(n, d) = \alpha(\Gamma_{n,d})$, the size of the largest independent set in $\Gamma_{n,d}$. This graph has $N = 2^n$ vertices and max degree $\Delta = 2^d \binom{n}{d}^2 \leq n^{2d}$.

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If Γ is a graph with N vertices, maximum degree Δ, and $O(ND^{2-\varepsilon})$ triangles, then

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Thus, it suffices to show that the number of triangles in $\Gamma_{n,d}$ is $O(2^n n^{4d-\varepsilon})$.

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Proof Sketch ($d = 1$)

Let u be a uniform random element of $V(\Gamma_{n,1}) = \{0,1\}^n$, and let v, w be two i.i.d. uniform random neighbors of u. We show that $\Pr[v \sim w] = O(\log n / n)$.

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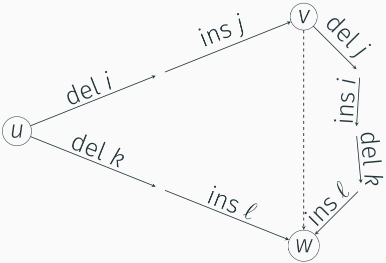

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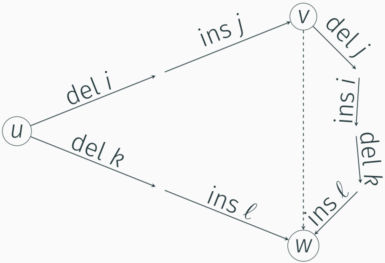

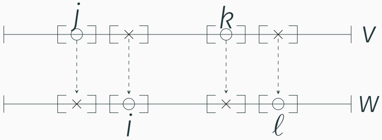

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- Same technique was used by Jiang and Vardy '04 to improve the Gilbert-Varshamov bound for bitflip errors by a logarithmic factor.
- Uses a **strong pseudorandomness property** of random strings u, v, w: every $\log n$-length subinterval is unique.

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Theorem (Guruswami, H., Li '22)

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Proof Strategy

Classify strings according to how much they "look like" $1^\ell 0^\ell 1^\ell 0^\ell \cdots$ for each power of two ℓ. A crude analogy is assigning $\log n$ "Fourier coefficients" to s that measure its oscillation on each scale.

Pigeonhole to find s, t with the same oscillation statistics. This guarantees $\mathrm{LCS}(s,t)$ is large for three possible reasons:

(1) If s, t oscillate at a large scale $\ell = \Omega(n)$.

(2) If s, t share at least one "large Fourier coefficient" at the same scale.

(3) If s, t share many "small Fourier coefficients" at different scales.

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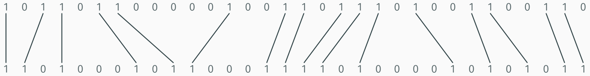
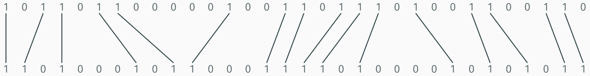
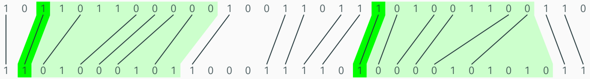


Matching Strategies

Case 1: Imbalanced Strings



Case 2: Single-Frequency Strings



Flagged bits	
Blue	$d(I) \geq 0.99$
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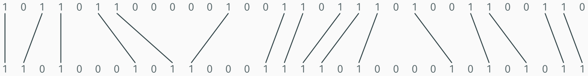

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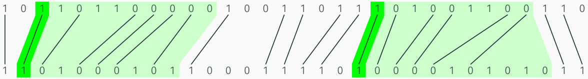

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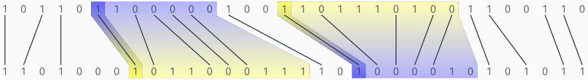

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Technical Difficulty

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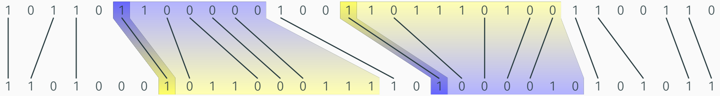

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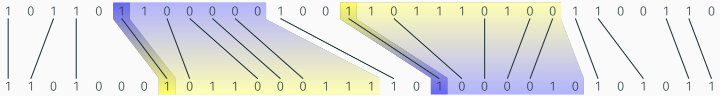

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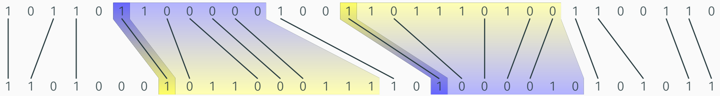

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The regularity method comes to the rescue!

The Regularity Method

Theorem (Szemerédi '78)

For every $\varepsilon > 0$, all sufficiently large graphs G can be partitioned into $O_\varepsilon(1)$ (nearly-)equal-sized vertex sets such that all but an ε-fraction of these pairs are ε-regular.

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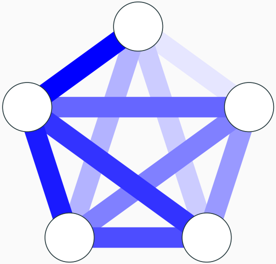

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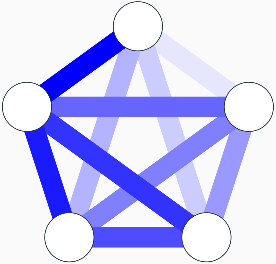

A pair of vertex sets X and Y are ε-regular if for all subsets $A \subseteq X$ and $B \subseteq Y$ satisfying $|A| \geq \varepsilon|X|$ and $|B| \geq \varepsilon|Y|$, we have $|d(A,B) - d(X,Y)| < \varepsilon$.

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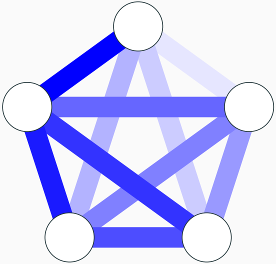

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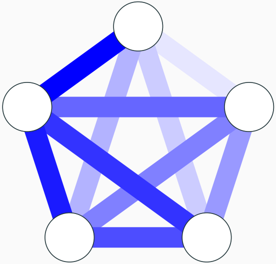

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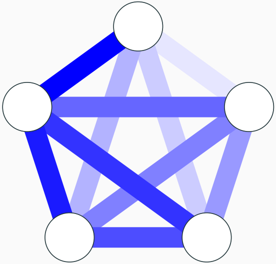

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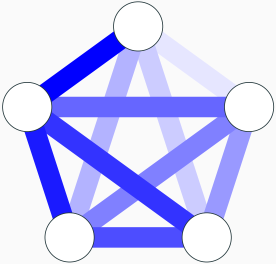

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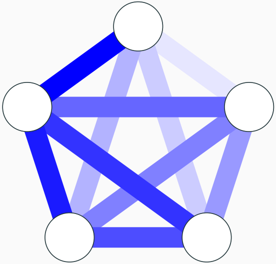

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- Connections to dynamics starting from the work of Furstenberg.
- Notorious for horrible quantitative bounds.

Regularity Lemmas for Binary Strings

Lemma (Axenovich, Person, Puzynina '12)

For every $\varepsilon > 0$ all sufficiently long binary strings s can be partitioned into $2^{\varepsilon^{-c}}$ (nearly-)equal-sized subintervals such that all but an ε-fraction of these subintervals are ε-regular.

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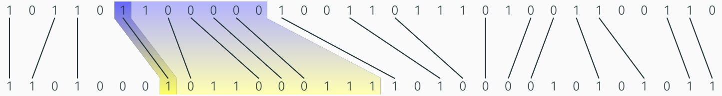

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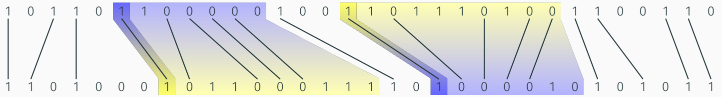

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- Fuzzy question: graph regularity leads to graphons. Is there a useful theory of the limit objects coming from string regularity?

Epilogue

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Uses the same “oscillation statistics” machinery.