Lower bounds for online Ramsey numbers

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Joint work with David Conlon, Jacob Fox, and Andrey Grinshpun

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Background

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Ramsey's Theorem (infinitary version)

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- The *online Ramsey number* $\tilde{r}(n)$ is the smallest number of edges of $K_{\mathbb{N}}$ one must reveal, **one at a time**, to find a monochromatic K_n.

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The best upper bounds on $r(n)$ are all proved using the Erdős-Szekeres method.

- $r(n) \leq \binom{2n-2}{n-1}$ (Erdős-Szekeres '35).
- $r(n) \leq n^{-\frac{C \log n}{\log \log n}} \cdot 2^{2n}$ (Conlon '09).

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Folklore bounds

$$2^{n/2} \leq \frac{r(n)}{2} \leq \tilde{r}(n) \leq \min\left\{2^{2n}, \binom{r(n)}{2}\right\}.$$

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Theorem (Conlon '09)

There exists $0 < c < 1$ *such that for infinitely many* n*,*

$$\tilde{r}(n) \leq c^n \binom{r(n)}{2}.$$

Main results

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Theorem (Conlon, Fox, Grinshpun, H. '19)

$$\tilde{r}(n) \geq 2^{(2-\sqrt{2})n - O(1)} \approx 2^{0.586n}.$$

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- Random painter.
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Main Ideas

- Random painter.
- Method of conditional expectation.
- Restrict to sets with large matchings.

Sketch of Proof

The Method of Conditional Expectation (Erdős-Selfridge '73)

- For each n-set $U \subseteq V(K_{\mathbb{N}})$, define

$$w(U,t) := \begin{cases} \left(\frac{1}{2}\right)^{\binom{n}{2}-e(U)} & U \text{ is monochromatic red at time } t \\ 0 & \text{otherwise.} \end{cases}$$

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- Suppose Builder plays the game for a total of N timesteps. We may pretend there are only $2N$ vertices.
- Define the total weight function

$$w(t) := \sum_{U \subseteq K_{2N}} w(U,t).$$

This function is a martingale in t.

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- This proves that $\tilde{r}(n) \geq 2^{n/2}$.

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If H is a graph with n vertices which contains a k-matching, and G has N edges, then the number of copies of H in G is $O(N^{n-k})$.

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- Optimizing $k = (1 - \frac{1}{\sqrt{2}})n$ completes the proof that

$$\tilde{r}(n) \geq 2^{(2-\sqrt{2})n-O(1)}.$$

The Subgraph Query Problem

Definition

Suppose we are given a finite graph H and want to find a copy of H in a hidden infinite random graph $G(\mathbb{N}, p)$, $p \in (0, 1)$. Let $f(H, p)$ be the number of adjacencies one must reveal, one at a time, to find a copy of H with probability $\frac{1}{2}$.

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Related work:

Theorem (Ferber, Krivelevich, Sudakov, Vieira '16)

Whenever $p > \frac{\log n + \log\log n + \omega(1)}{n}$*, one can find (w.h.p.) a Hamiltonian cycle in a hidden* $G(n, p)$ *by revealing at most* $(1 + o(1))n$ **edges**.

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Theorem (Feige, Gamarnik, Neeman, Rácz, Tetali '18)

For any $\delta < 2$, *there is an* $\alpha < 2$ *such that one cannot find (w.c.p.) a clique of order* $\alpha \log_2 n$ *in a hidden* $G(n, \frac{1}{2})$ *by revealing at most* n^{δ} *adjacencies using a constant number of rounds of adaptiveness.*

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By definition, Builder has a strategy that guarantees a win against a random Painter in $\tilde{r}(n)$ moves. In particular, this either this strategy builds a red K_n with probability $\frac{1}{2}$ or a blue K_n with probability $\frac{1}{2}$. □

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Theorem (Conlon, Fox, Grinshpun, H. '19)

For any fixed n, as $p \to 0^+$,

$$p^{-(2-\sqrt{2})n+O(1)} \leq f(K_n, p) \leq p^{-\frac{2}{3}n-O(1)}.$$

The Subgraph Query Problem

Theorem (Conlon, Fox, Grinshpun, H. '19)

As $p \to 0^+$,

$$\begin{aligned} f(K_3, p) &= \Theta(p^{-\frac{3}{2}}) \\ f(K_4, p) &= \Theta(p^{-2}) \\ f(K_5, p) &= \Theta(p^{-\frac{8}{3}}). \end{aligned}$$

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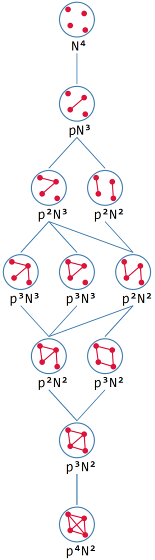

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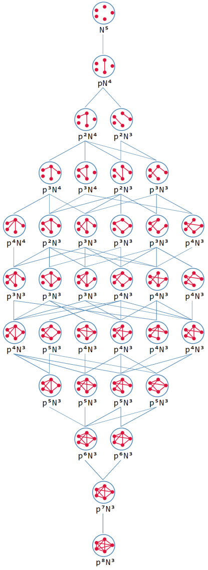

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Theorem (Conlon, Fox, Grinshpun, H. '19)

For any $m \ge 3$, *as* $n \to \infty$,

$$n^{(2-\sqrt{2})m - O(1)} \le \tilde{r}(m,n) \le O_m\left(\frac{n^m}{(\log n)^{\lfloor m/2 \rfloor - 1}}\right).$$

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Conjecture

For every $d \geq 2$*, there exists a d-degenerate graph H for which*

$$f(H, p) = \Theta(p^{-d})$$

as $p \to 0^+$*.*