Ramsey numbers of sparse digraphs

Xiaoyu He (Stanford)

Joint with Jacob Fox and Yuval Wigderson

March 9, 2021

Warmup: Hamiltonian paths in tournaments

Theorem (Rédei 1934)

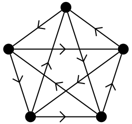
Every tournament contains a Hamiltonian path.

Warmup: Hamiltonian paths in tournaments

Theorem (Rédei 1934)

Every tournament contains a Hamiltonian path.

Proof: Take a *median ordering* of a tournament, which orders the vertices to maximize the number of forward edges.

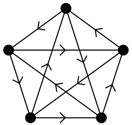


Warmup: Hamiltonian paths in tournaments

Theorem (Rédei 1934)

Every tournament contains a Hamiltonian path.

Proof: Take a *median ordering* of a tournament, which orders the vertices to maximize the number of forward edges.



Definition

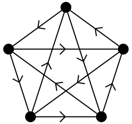
The (oriented) *Ramsey number* $\vec{r}(H)$ of a digraph H is the minimum N such that every N-vertex tournament contains a copy of H.

Warmup: Hamiltonian paths in tournaments

Theorem (Rédei 1934)

Every tournament contains a Hamiltonian path.

Proof: Take a *median ordering* of a tournament, which orders the vertices to maximize the number of forward edges.



Definition

The (oriented) *Ramsey number* $\vec{r}(H)$ of a digraph H is the minimum N such that every N-vertex tournament contains a copy of H.

Rédei's theorem is equivalent to $\vec{r}(P_n) = n$, where P_n is the oriented path on n vertices.

Undirected Ramsey numbers

Definition

The *Ramsey number* $\vec{r}(H)$ of a digraph H is the minimum N such that every N-vertex tournament contains a copy of H.

Undirected Ramsey numbers

Definition

The *Ramsey number* $r(H)$ of a graph H is the minimum N such that every N-vertex two-edge-colored complete graph contains a monochromatic copy of H.

Undirected Ramsey numbers

Definition

The *Ramsey number* $r(H)$ of a graph H is the minimum N such that every N-vertex two-edge-colored complete graph contains a monochromatic copy of H.

Theorem (Erdős–Szekeres 1935, Erdős 1947)

$$2^{n/2} \leq r(K_n) \leq 2^{2n}$$

Undirected Ramsey numbers

Definition

The *Ramsey number* $r(H)$ of a graph H is the minimum N such that every N-vertex two-edge-colored complete graph contains a monochromatic copy of H.

Theorem (Erdős–Szekeres 1935, Erdős 1947)

$$2^{n/2} \le r(K_n) \le 2^{2n}$$

- The upper bound implies that $r(H)$ exists for all H (Ramsey 1930).

Undirected Ramsey numbers

Definition

The *Ramsey number* $r(H)$ of a graph H is the minimum N such that every N-vertex two-edge-colored complete graph contains a monochromatic copy of H.

Theorem (Erdős–Szekeres 1935, Erdős 1947)

$$2^{n/2} \leq r(K_n) \leq 2^{2n}$$

- The upper bound implies that $r(H)$ exists for all H (Ramsey 1930).
- In general, $r(H)$ can be exponential in n, but if H is a tree or cycle, then $r(H) = O(n)$.

Undirected Ramsey numbers

Definition

The *Ramsey number* $r(H)$ of a graph H is the minimum N such that every N-vertex two-edge-colored complete graph contains a monochromatic copy of H.

Theorem (Erdős–Szekeres 1935, Erdős 1947)

$$2^{n/2} \leq r(K_n) \leq 2^{2n}$$

- The upper bound implies that $r(H)$ exists for all H (Ramsey 1930).
- In general, $r(H)$ can be exponential in n, but if H is a tree or cycle, then $r(H) = O(n)$.
- Burr–Erdős (1975): Is $r(H)$ linear in n for all sparse graphs H?

The Burr–Erdős conjecture

Burr–Erdős (1975): Is $r(H)$ linear in n for all sparse graphs H?

Theorem (Chvátal–Rödl–Szemerédi–Trotter 1983)

If H has n vertices and maximum degree Δ, then $r(H) = O_\Delta(n)$.

The Burr–Erdős conjecture

Burr–Erdős (1975): Is $r(H)$ linear in n for all sparse graphs H?

Theorem (Chvátal–Rödl–Szemerédi–Trotter 1983)

If H has n vertices and maximum degree Δ, then $r(H) = O_\Delta(n)$.

A weaker notion of sparsity is degeneracy. The *degeneracy* of H is

$$d := \max_{H' \subseteq H} (\mathrm{mindeg}(H')).$$

The Burr–Erdős conjecture

Burr–Erdős (1975): Is $r(H)$ linear in n for all sparse graphs H?

Theorem (Chvátal–Rödl–Szemerédi–Trotter 1983)

If H has n vertices and maximum degree Δ, then $r(H) = O_\Delta(n)$.

A weaker notion of sparsity is degeneracy. The *degeneracy* of H is

$$d := \max_{H' \subseteq H}(\text{mindeg}(H')).$$

If H is d-degenerate, then $r(H) \geq 2^{d/2}$, so graphs of high degeneracy have large Ramsey numbers.

The Burr–Erdős conjecture

Burr–Erdős (1975): Is $r(H)$ linear in n for all sparse graphs H?

Theorem (Chvátal–Rödl–Szemerédi–Trotter 1983)

If H has n vertices and maximum degree Δ, then $r(H) = O_\Delta(n)$.

A weaker notion of sparsity is degeneracy. The *degeneracy* of H is

$$d := \max_{H' \subseteq H}(\text{mindeg}(H')).$$

If H is d-degenerate, then $r(H) \geq 2^{d/2}$, so graphs of high degeneracy have large Ramsey numbers.

Conjecture (Burr–Erdős 1975)

If H is d-degenerate, then $r(H) = O_d(n)$.

The Burr–Erdős conjecture

Burr–Erdős (1975): Is $r(H)$ linear in n for all sparse graphs H?

Theorem (Chvátal–Rödl–Szemerédi–Trotter 1983)

If H has n vertices and maximum degree Δ, then $r(H) = O_\Delta(n)$.

A weaker notion of sparsity is degeneracy. The *degeneracy* of H is

$$d := \max_{H' \subseteq H}(\text{mindeg}(H')).$$

If H is d-degenerate, then $r(H) \geq 2^{d/2}$, so graphs of high degeneracy have large Ramsey numbers.

Conjecture (Burr–Erdős 1975), Theorem (Lee 2017)

If H is d-degenerate, then $r(H) = O_d(n)$.

The Burr–Erdős conjecture

Burr–Erdős (1975): Is $r(H)$ linear in n for all sparse graphs H?

Theorem (Chvátal–Rödl–Szemerédi–Trotter 1983)

If H has n vertices and maximum degree Δ, then $r(H) = O_\Delta(n)$.

A weaker notion of sparsity is degeneracy. The *degeneracy* of H is

$$d := \max_{H' \subseteq H}(\text{mindeg}(H')).$$

If H is d-degenerate, then $r(H) \geq 2^{d/2}$, so graphs of high degeneracy have large Ramsey numbers.

Conjecture (Burr–Erdős 1975), Theorem (Lee 2017)

If H is d-degenerate, then $r(H) = O_d(n)$.

Upshot: H has linear Ramsey number "if and only if" H is sparse.
Qualitatively, n and d control $r(H)$.

Ramsey numbers of digraphs

Definition

The *Ramsey number* $\vec{r}(H)$ of a digraph H is the minimum N such that every N-vertex tournament contains a copy of H.

Ramsey numbers of digraphs

Definition

The *Ramsey number* $\vec{r}(H)$ of a digraph H is the minimum N such that every N-vertex tournament contains a copy of H.

Observation: A transitive tournament contains only acyclic digraphs. So $\vec{r}(H) = \infty$ unless H is acyclic.

Ramsey numbers of digraphs

Definition

The *Ramsey number* $\vec{r}(H)$ of a digraph H is the minimum N such that every N-vertex tournament contains a copy of H.

Observation: A transitive tournament contains only acyclic digraphs. So $\vec{r}(H) = \infty$ unless H is acyclic.

Theorem (Stearns 1959, Erdős–Moser 1964)

If $\overrightarrow{T_n}$ denotes the transitive tournament on n vertices, then

$$2^{n/2-1} \leq \vec{r}(\overrightarrow{T_n}) \leq 2^{n-1}.$$

Ramsey numbers of digraphs

Definition

The *Ramsey number* $\vec{r}(H)$ of a digraph H is the minimum N such that every N-vertex tournament contains a copy of H.

Observation: A transitive tournament contains only acyclic digraphs. So $\vec{r}(H) = \infty$ unless H is acyclic.

Theorem (Stearns 1959, Erdős–Moser 1964)

If $\overrightarrow{T_n}$ denotes the transitive tournament on n vertices, then

$$2^{n/2-1} \leq \vec{r}(\overrightarrow{T_n}) \leq 2^{n-1}.$$

- The upper bound implies that $\vec{r}(H)$ exists for all acyclic H. Like the undirected Ramsey number, $\vec{r}(H)$ can be exponential in n.

Ramsey numbers of digraphs

Definition

The *Ramsey number* $\vec{r}(H)$ of a digraph H is the minimum N such that every N-vertex tournament contains a copy of H.

Observation: A transitive tournament contains only acyclic digraphs. So $\vec{r}(H) = \infty$ unless H is acyclic.

Theorem (Stearns 1959, Erdős–Moser 1964)

If $\overrightarrow{T_n}$ denotes the transitive tournament on n vertices, then

$$2^{n/2-1} \leq \vec{r}(\overrightarrow{T_n}) \leq 2^{n-1}.$$

- The upper bound implies that $\vec{r}(H)$ exists for all acyclic H. Like the undirected Ramsey number, $\vec{r}(H)$ can be exponential in n.
- Is the analogue of the Burr-Erdős conjecture true for digraphs?

Is $\vec{r}(H) = O(n)$ for sparse H?

Main Question (Bucić–Letzter–Sudakov 2019)

Is $\vec{r}(H) = O(n)$ for all bounded-degree H?

Is $\vec{r}(H) = O(n)$ for sparse H?

Main Question (Bucić–Letzter–Sudakov 2019)

Is $\vec{r}(H) = O(n)$ for all bounded-degree H?

Conjecture (Sumner 1971)

If H is some orientation of an n-vertex tree, then $\vec{r}(H) \leq 2n - 2$.

Is $\vec{r}(H) = O(n)$ for sparse H?

Main Question (Bucić–Letzter–Sudakov 2019)

Is $\vec{r}(H) = O(n)$ for all bounded-degree H?

Conjecture (Sumner 1971)

If H is some orientation of an n-vertex tree, then $\vec{r}(H) \leq 2n - 2$.

Progress on Sumner's conjecture

- Häggkvist–Thomason (1991): $\vec{r}(H) \leq 12n$.

Is $\vec{r}(H) = O(n)$ for sparse H?

Main Question (Bucić–Letzter–Sudakov 2019)

Is $\vec{r}(H) = O(n)$ for all bounded-degree H?

Conjecture (Sumner 1971)

If H is some orientation of an n-vertex tree, then $\vec{r}(H) \leq 2n - 2$.

Progress on Sumner's conjecture

- Häggkvist–Thomason (1991): $\vec{r}(H) \leq 12n$.
- Kühn–Mycroft–Osthus (2011): $\vec{r}(H) \leq 2n - 2$ for n large enough.

Is $\vec{r}(H) = O(n)$ for sparse H?

Main Question (Bucić–Letzter–Sudakov 2019)

Is $\vec{r}(H) = O(n)$ for all bounded-degree H?

Conjecture (Sumner 1971)

If H is some orientation of an n-vertex tree, then $\vec{r}(H) \leq 2n - 2$.

Progress on Sumner's conjecture

- Häggkvist–Thomason (1991): $\vec{r}(H) \leq 12n$.
- Kühn–Mycroft–Osthus (2011): $\vec{r}(H) \leq 2n - 2$ for n large enough.
- Dross–Havet (2019): $\vec{r}(H) \leq \frac{21}{8}n - \frac{47}{16}$.

Is $\vec{r}(H) = O(n)$ for sparse H?

Main Question (Bucić–Letzter–Sudakov 2019)

Is $\vec{r}(H) = O(n)$ for all bounded-degree H?

Conjecture (Sumner 1971)

If H is some orientation of an n-vertex tree, then $\vec{r}(H) \leq 2n - 2$.

Progress on Sumner's conjecture

- Häggkvist–Thomason (1991): $\vec{r}(H) \leq 12n$.
- Kühn–Mycroft–Osthus (2011): $\vec{r}(H) \leq 2n - 2$ for n large enough.
- Dross–Havet (2019): $\vec{r}(H) \leq \frac{21}{8}n - \frac{47}{16}$.

What about other sparse digraphs H?

Is $\vec{r}(H) = O(n)$ for sparse H?

Main Question (Bucić–Letzter–Sudakov 2019)

Is $\vec{r}(H) = O(n)$ for all bounded-degree H?

Is $\vec{r}(H) = O(n)$ for sparse H?

Main Question (Bucić–Letzter–Sudakov 2019)

Is $\vec{r}(H) = O(n)$ for all bounded-degree H?

Theorem (Thomason 1986)

If H is any acyclic orientation of C_n, then $\vec{r}(H) = n$ for n large enough.

Is $\vec{r}(H) = O(n)$ for sparse H?

Main Question (Bucić–Letzter–Sudakov 2019)

Is $\vec{r}(H) = O(n)$ for all bounded-degree H?

Theorem (Thomason 1986)

If H is any acyclic orientation of C_n, then $\vec{r}(H) = n$ for n large enough.

Define the *bandwidth* of an acyclic digraph H to be the minimum k such that H can be ordered $v_1, \ldots, v_n$ so that edges $v_i \to v_j$ exist only if $1 \leq j - i \leq k$.

Is $\vec{r}(H) = O(n)$ for sparse H?

Main Question (Bucić–Letzter–Sudakov 2019)

Is $\vec{r}(H) = O(n)$ for all bounded-degree H?

Theorem (Thomason 1986)

If H is any acyclic orientation of C_n, then $\vec{r}(H) = n$ for n large enough.

Define the *bandwidth* of an acyclic digraph H to be the minimum k such that H can be ordered $v_1, \ldots, v_n$ so that edges $v_i \to v_j$ exist only if $1 \leq j - i \leq k$.

Theorem (DDFGHKLMSS 2020)

If H has bandwidth k, then $\vec{r}(H) = O_k(n)$.

Our Results

Theorem (Fox–H.–Wigderson 2021)

For all $C > 0$ and n large enough, there is an n-vertex acyclic digraph H with maximum degree $\Delta \leq C^{3/2+o(1)}$ satisfying $\vec{r}(H) > n^C$.

Our Results

Theorem (Fox–H.–Wigderson 2021)

For all $C > 0$ and n large enough, there is an n-vertex acyclic digraph H with maximum degree $\Delta \leq C^{3/2+o(1)}$ satisfying $\vec{r}(H) > n^C$.

Theorem (Fox–H.–Wigderson 2021)

Let H be an n-vertex acyclic digraph with maximum degree Δ.

Our Results

Theorem (Fox–H.–Wigderson 2021)

For all $C > 0$ *and* n *large enough, there is an* n*-vertex acyclic digraph* H *with maximum degree* $\Delta \leq C^{3/2+o(1)}$ *satisfying* $\vec{r}(H) > n^C$.

Theorem (Fox–H.–Wigderson 2021)

Let H *be an* n*-vertex acyclic digraph with maximum degree* Δ.

- *In general,* $\vec{r}(H) \leq n^{O_\Delta(\log n)}$.

Our Results

Theorem (Fox–H.–Wigderson 2021)

For all $C > 0$ and n large enough, there is an n-vertex acyclic digraph H with maximum degree $\Delta \leq C^{3/2+o(1)}$ satisfying $\vec{r}(H) > n^C$.

Theorem (Fox–H.–Wigderson 2021)

Let H be an n-vertex acyclic digraph with maximum degree Δ.

- *In general, $\vec{r}(H) \leq n^{O_\Delta(\log n)}$.*
- *If H is chosen randomly, then $\vec{r}(H) \leq n \cdot (\log n)^{O_\Delta(1)}$.*

Our Results

Theorem (Fox–H.–Wigderson 2021)

For all $C > 0$ and n large enough, there is an n-vertex acyclic digraph H with maximum degree $\Delta \le C^{3/2+o(1)}$ satisfying $\vec{r}(H) > n^C$.

Theorem (Fox–H.–Wigderson 2021)

Let H be an n-vertex acyclic digraph with maximum degree Δ.

- *In general, $\vec{r}(H) \le n^{O_\Delta(\log n)}$.*
- *If H is chosen randomly, then $\vec{r}(H) \le n \cdot (\log n)^{O_\Delta(1)}$.*
- *If H has height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.*

Our Results

Theorem (Fox–H.–Wigderson 2021)

For all $C > 0$ and n large enough, there is an n-vertex acyclic digraph H with maximum degree $\Delta \leq C^{3/2+o(1)}$ satisfying $\vec{r}(H) > n^C$.

Theorem (Fox–H.–Wigderson 2021)

Let H be an n-vertex acyclic digraph with maximum degree Δ.

- *In general, $\vec{r}(H) \leq n^{O_\Delta(\log n)}$.*
- *If H is chosen randomly, then $\vec{r}(H) \leq n \cdot (\log n)^{O_\Delta(1)}$.*
- *If H has height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.*

Our Results

Theorem (Fox–H.–Wigderson 2021)

For all $C > 0$ and n large enough, there is an n-vertex acyclic digraph H with maximum degree $\Delta \leq C^{3/2+o(1)}$ satisfying $\vec{r}(H) > n^C$.

Theorem (Fox–H.–Wigderson 2021)

Let H be an n-vertex acyclic digraph with maximum degree Δ.

- *In general, $\vec{r}(H) \leq n^{O_\Delta(\log n)}$.*
- *If H is chosen randomly, then $\vec{r}(H) \leq n \cdot (\log n)^{O_\Delta(1)}$.*
- *If H has height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.*

Outline of Talk

1. Proof of lower bound $\vec{r}(H) > n^{\log_2(3)-o(1)}$.

Our Results

Theorem (Fox–H.–Wigderson 2021)

For all $C > 0$ and n large enough, there is an n-vertex acyclic digraph H with maximum degree $\Delta \leq C^{3/2+o(1)}$ satisfying $\vec{r}(H) > n^C$.

Theorem (Fox–H.–Wigderson 2021)

Let H be an n-vertex acyclic digraph with maximum degree Δ.

- *In general, $\vec{r}(H) \leq n^{O_\Delta(\log n)}$.*
- *If H is chosen randomly, then $\vec{r}(H) \leq n \cdot (\log n)^{O_\Delta(1)}$.*
- *If H has height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.*

Outline of Talk

1. Proof of lower bound $\vec{r}(H) > n^{\log_2(3)-o(1)}$.
2. Proof of the height upper bound: greedy embedding.

Our Results

Theorem (Fox–H.–Wigderson 2021)

For all $C > 0$ and n large enough, there is an n-vertex acyclic digraph H with maximum degree $\Delta \leq C^{3/2+o(1)}$ satisfying $\vec{r}(H) > n^C$.

Theorem (Fox–H.–Wigderson 2021)

Let H be an n-vertex acyclic digraph with maximum degree Δ.

- *In general, $\vec{r}(H) \leq n^{O_\Delta(\log n)}$.*
- *If H is chosen randomly, then $\vec{r}(H) \leq n \cdot (\log n)^{O_\Delta(1)}$.*
- *If H has height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.*

Outline of Talk

1. Proof of lower bound $\vec{r}(H) > n^{\log_2(3)-o(1)}$.
2. Proof of the height upper bound: greedy embedding.
3. Results on multicolor Ramsey numbers.

Lower bound proof

Theorem

There exists an n-vertex acyclic digraph H with maximum degree 1000 *and* $\vec{r}(H) > n^{\log_2(3)-o(1)}$.

Lower bound proof

Theorem

There exists an n-vertex acyclic digraph H with maximum degree 1000 *and* $\vec{r}(H) > n^{\log_2(3)-o(1)}$.

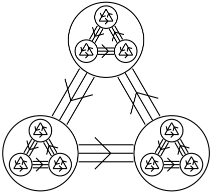
Let T be an iterated blowup of a cyclic triangle on N vertices:

Lower bound proof

Theorem

There exists an n-vertex acyclic digraph H with maximum degree 1000 *and* $\vec{r}(H) > n^{\log_2(3)-o(1)}$.

Let T be an iterated blowup of a cyclic triangle on N vertices:

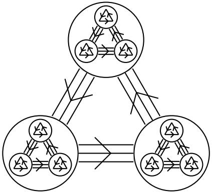


Lower bound proof

Theorem

There exists an n-vertex acyclic digraph H with maximum degree 1000 *and* $\vec{r}(H) > n^{\log_2(3)-o(1)}$.

Let T be an iterated blowup of a cyclic triangle on N vertices:



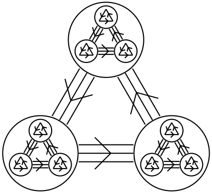
Key Idea: We will construct H on $[n]$ so that in any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $0.49n$ maps to a single third of T.

Lower bound proof

Theorem

There exists an n-vertex acyclic digraph H with maximum degree 1000 *and* $\vec{r}(H) > n^{\log_2(3)-o(1)}$.

Let T be an iterated blowup of a cyclic triangle on N vertices:



Key Idea: We will construct H on $[n]$ so that in any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $0.49n$ maps to a single third of T.

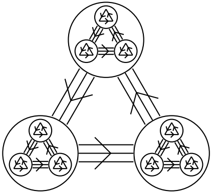
We ensure that induced subgraphs of H on subintervals inherit this property, so we can iterate.

Lower bound proof

Theorem

There exists an n-vertex acyclic digraph H with maximum degree 1000 *and* $\vec{r}(H) > n^{\log_2(3)-o(1)}$.

Let T be an iterated blowup of a cyclic triangle on N vertices:



Key Idea: We will construct H on $[n]$ so that in any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $0.49n$ maps to a single third of T.

We ensure that induced subgraphs of H on subintervals inherit this property, so we can iterate. After t iterations, we find a subinterval of H of length $(0.49)^t n$ that maps to a single part of T of size $3^{-t}N$.

Lower bound proof: interval meshes

Want: In any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $0.49n$ is mapped into a single part, and this property is hereditary.

Lower bound proof: interval meshes

Want: In any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $0.49n$ is mapped into a single part, and this property is hereditary.

Definition

A digraph H on $[n]$ is an *interval mesh* if edges point forward and:

Lower bound proof: interval meshes

Want: In any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $0.49n$ is mapped into a single part, and this property is hereditary.

Definition

A digraph H on $[n]$ is an *interval mesh* if edges point forward and:

For all $0 \le a < b \le c < d \le n$ with $c - b \le 100 \min(b - a, d - c)$, there is an edge from $(a, b]$ to $(c, d]$.

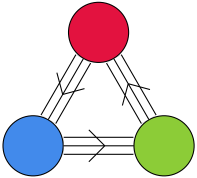
Lower bound proof: interval meshes

Want: In any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $0.49n$ is mapped into a single part, and this property is hereditary.

Definition

A digraph H on $[n]$ is an *interval mesh* if edges point forward and:

For all $0 \le a < b \le c < d \le n$ with $c - b \le 100 \min(b - a, d - c)$, there is an edge from $(a, b]$ to $(c, d]$.



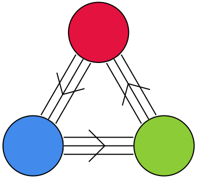
Lower bound proof: interval meshes

Want: In any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $0.49n$ is mapped into a single part, and this property is hereditary.

Definition

A digraph H on $[n]$ is an *interval mesh* if edges point forward and:

For all $0 \leq a < b \leq c < d \leq n$ with $c - b \leq 100 \min(b - a, d - c)$, there is an edge from $(a, b]$ to $(c, d]$.



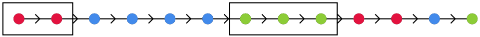
Lower bound proof: interval meshes

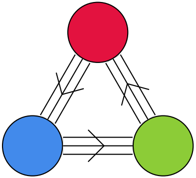
Want: In any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $0.49n$ is mapped into a single part, and this property is hereditary.

Definition

A digraph H on $[n]$ is an *interval mesh* if edges point forward and:

For all $0 \leq a < b \leq c < d \leq n$ with $c - b \leq 100 \min(b - a, d - c)$, there is an edge from $(a, b]$ to $(c, d]$.





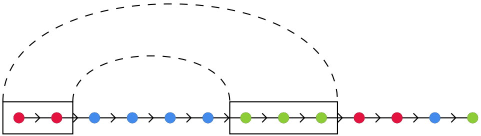
Lower bound proof: interval meshes

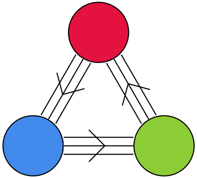
Want: In any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $0.49n$ is mapped into a single part, and this property is hereditary.

Definition

A digraph H on $[n]$ is an *interval mesh* if edges point forward and:

For all $0 \leq a < b \leq c < d \leq n$ with $c - b \leq 100 \min(b - a, d - c)$, there is an edge from $(a, b]$ to $(c, d]$.





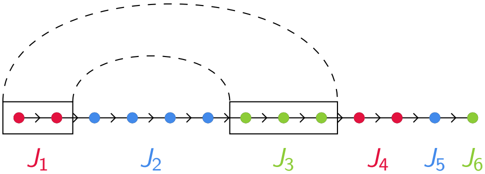
Lower bound proof: interval meshes

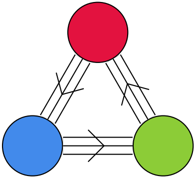
Want: In any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $0.49n$ is mapped into a single part, and this property is hereditary.

Definition

A digraph H on $[n]$ is an *interval mesh* if edges point forward and:

For all $0 \leq a < b \leq c < d \leq n$ with $c - b \leq 100 \min(b - a, d - c)$, there is an edge from $(a, b]$ to $(c, d]$.





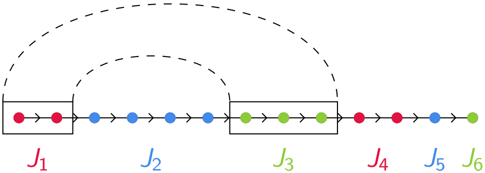
Lower bound proof: interval meshes

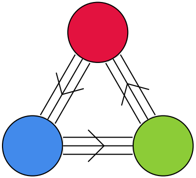
Want: In any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $0.49n$ is mapped into a single part, and this property is hereditary.

Definition

A digraph H on $[n]$ is an *interval mesh* if edges point forward and:

For all $0 \leq a < b \leq c < d \leq n$ with $c - b \leq 100 \min(b - a, d - c)$, there is an edge from $(a, b]$ to $(c, d]$.





Thus, $|J_i| > 100 \min(|J_{i-1}|, |J_{i+1}|)$.

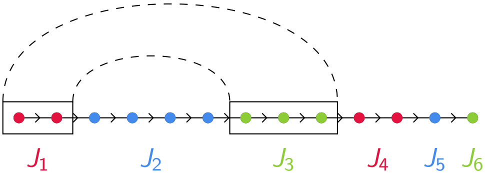
Lower bound proof: interval meshes

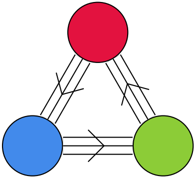
Want: In any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $0.49n$ is mapped into a single part, and this property is hereditary.

Definition

A digraph H on $[n]$ is an *interval mesh* if edges point forward and:

For all $0 \leq a < b \leq c < d \leq n$ with $c - b \leq 100 \min(b - a, d - c)$, there is an edge from $(a, b]$ to $(c, d]$.





Thus, $|J_i| > 100 \min(|J_{i-1}|, |J_{i+1}|)$. So $|J_i|$ is unimodal and $\max_i |J_i| \geq 0.49n$.

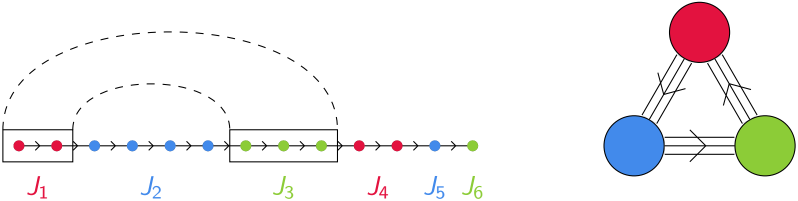
Lower bound proof: interval meshes

Want: In any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $0.49n$ is mapped into a single part, and this property is hereditary.

Definition

A digraph H on $[n]$ is an *interval mesh* if edges point forward and:

For all $0 \le a < b \le c < d \le n$ with $c - b \le 100 \min(b - a, d - c)$, there is an edge from $(a, b]$ to $(c, d]$.



Thus, $|J_i| > 100 \min(|J_{i-1}|, |J_{i+1}|)$. So $|J_i|$ is unimodal and $\max_i |J_i| \ge 0.49n$.

We can greedily construct interval meshes on $[n]$ with max degree 1000.

Upper bound proof: greedy embedding

Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Upper bound proof: greedy embedding

Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.

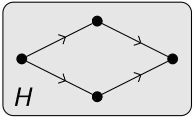
Upper bound proof: greedy embedding

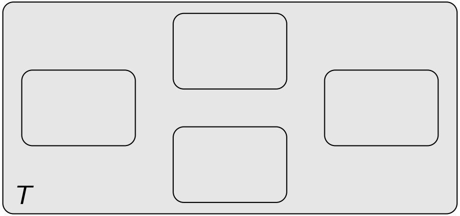
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.





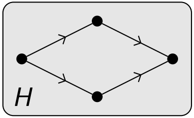
Upper bound proof: greedy embedding

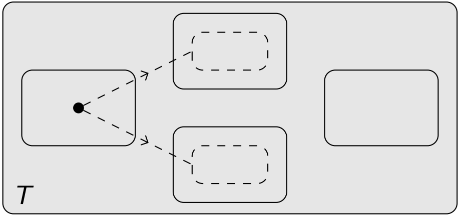
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.





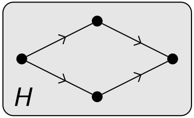
Upper bound proof: greedy embedding

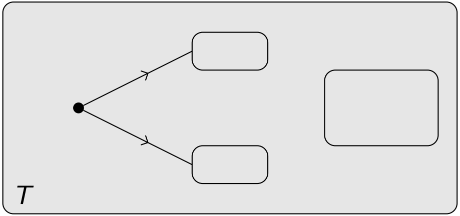
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.





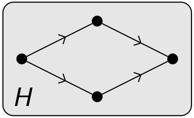
Upper bound proof: greedy embedding

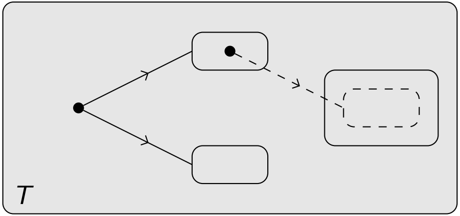
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.





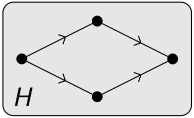
Upper bound proof: greedy embedding

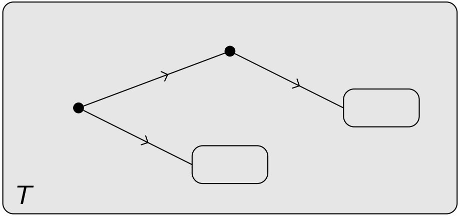
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.





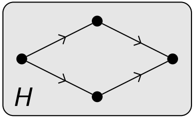
Upper bound proof: greedy embedding

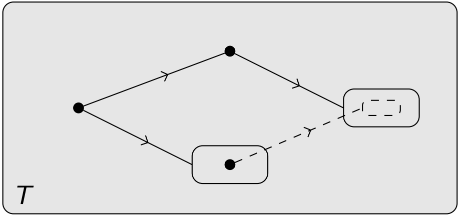
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.





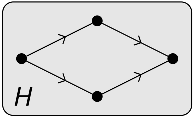
Upper bound proof: greedy embedding

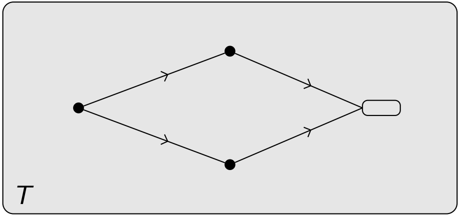
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.





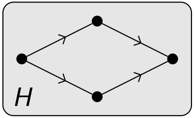
Upper bound proof: greedy embedding

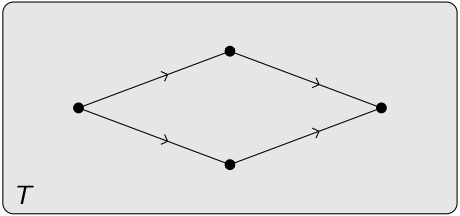
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.





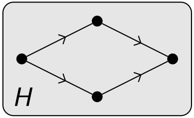
Upper bound proof: greedy embedding

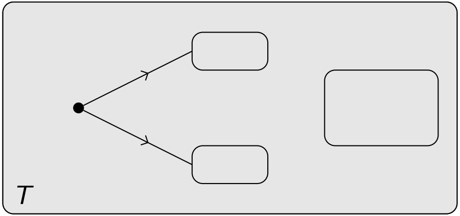
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.





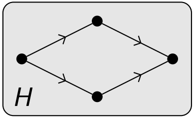
Upper bound proof: greedy embedding

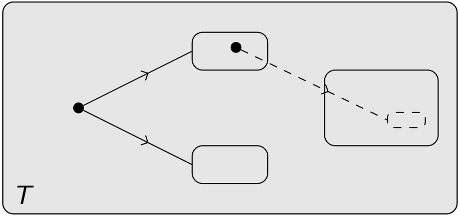
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.





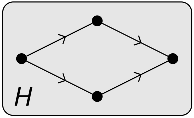
Upper bound proof: greedy embedding

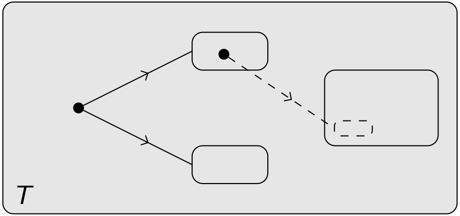
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.





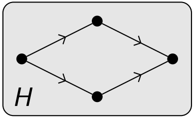
Upper bound proof: greedy embedding

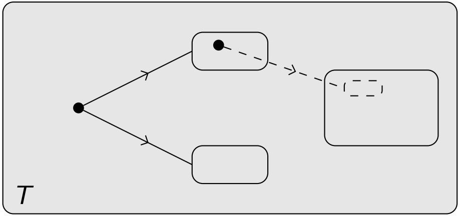
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.





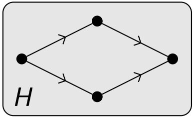
Upper bound proof: greedy embedding

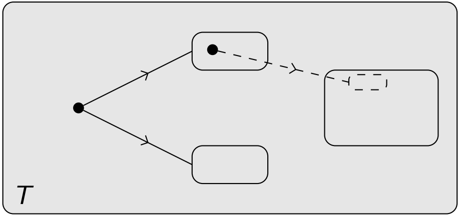
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.





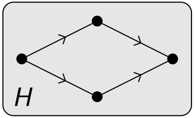
Upper bound proof: greedy embedding

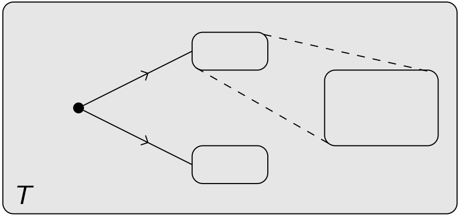
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.





Upper bound proof: greedy embedding

Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.

Upper bound proof: greedy embedding

Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.

Iterating the lemma $\log h$ times, we find h sets in T with most edges oriented forwards:

Upper bound proof: greedy embedding

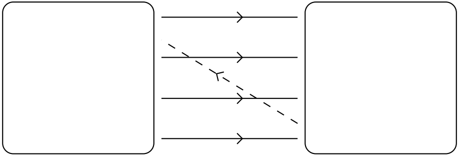
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.

Iterating the lemma $\log h$ times, we find h sets in T with most edges oriented forwards:



Upper bound proof: greedy embedding

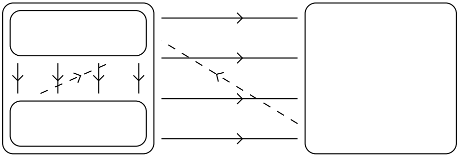
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.

Iterating the lemma $\log h$ times, we find h sets in T with most edges oriented forwards:



Upper bound proof: greedy embedding

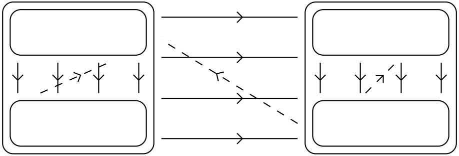
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.

Iterating the lemma $\log h$ times, we find h sets in T with most edges oriented forwards:



Upper bound proof: greedy embedding

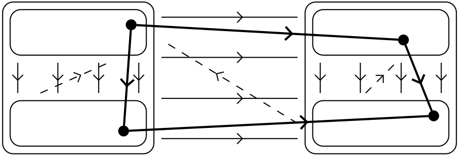
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.

Iterating the lemma $\log h$ times, we find h sets in T with most edges oriented forwards:



Upper bound proof: greedy embedding

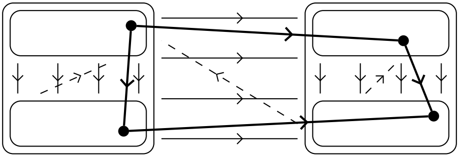
Theorem

If H has n vertices, max degree Δ, and height h, then $\vec{r}(H) = O_{\Delta,h}(n)$.

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.

Iterating the lemma $\log h$ times, we find h sets in T with most edges oriented forwards:



It is easy to embed H into these sets greedily.

What about more than one color?

Definition

If $k \geq 1$, the k-color Ramsey number $\overrightarrow{r_k}(H)$ of a digraph H is the minimum N such that any k-edge-colored N-vertex tournament contains a monochromatic copy of H.

What about more than one color?

Definition

If $k \geq 1$, the k-color Ramsey number $\overrightarrow{r_k}(H)$ of a digraph H is the minimum N such that any k-edge-colored N-vertex tournament contains a monochromatic copy of H.

We showed: For $k = 1$, if H has n vertices and maximum degree Δ, then $\overrightarrow{r_1}(H) \leq n^{O_\Delta(\log n)}$, but $\overrightarrow{r_1}(H) \geq n^C$ is possible for any $C > 0$.

What about more than one color?

Definition

If $k \geq 1$, the k-color Ramsey number $\overrightarrow{r_k}(H)$ of a digraph H is the minimum N such that any k-edge-colored N-vertex tournament contains a monochromatic copy of H.

We showed: For $k = 1$, if H has n vertices and maximum degree Δ, then $\overrightarrow{r_1}(H) \leq n^{O_\Delta(\log n)}$, but $\overrightarrow{r_1}(H) \geq n^C$ is possible for any $C > 0$.

Theorem (Fox–H.–Wigderson 2021)

If H has n vertices and maximum degree Δ, then

$$\overrightarrow{r_k}(H) \leq n^{O_\Delta(\log^{O_k(1)} n)}.$$

What about more than one color?

Definition

If $k \geq 1$, the k-color Ramsey number $\overrightarrow{r_k}(H)$ of a digraph H is the minimum N such that any k-edge-colored N-vertex tournament contains a monochromatic copy of H.

We showed: For $k = 1$, if H has n vertices and maximum degree Δ, then $\overrightarrow{r_1}(H) \leq n^{O_\Delta(\log n)}$, but $\overrightarrow{r_1}(H) \geq n^C$ is possible for any $C > 0$.

Theorem (Fox–H.–Wigderson 2021)

If H has n vertices and maximum degree Δ, then

$$\overrightarrow{r_k}(H) \leq n^{O_\Delta(\log^{O_k(1)} n)}.$$

For $k \geq 2$, there exists H of maximum degree 3 and

$$\overrightarrow{r_k}(H) \geq n^{\Omega(\log n / \log \log n)}.$$

What about more than one color?

Definition

If $k \geq 1$, the k-color Ramsey number $\overrightarrow{r_k}(H)$ of a digraph H is the minimum N such that any k-edge-colored N-vertex tournament contains a monochromatic copy of H.

We showed: For $k = 1$, if H has n vertices and maximum degree Δ, then $\overrightarrow{r_1}(H) \leq n^{O_\Delta(\log n)}$, but $\overrightarrow{r_1}(H) \geq n^C$ is possible for any $C > 0$.

Theorem (Fox–H.–Wigderson 2021)

If H has n vertices and maximum degree Δ, then

$$\overrightarrow{r_k}(H) \leq n^{O_\Delta(\log^{O_k(1)} n)}.$$

For $k \geq 2$, there exists H of maximum degree 3 and

$$\overrightarrow{r_k}(H) \geq n^{\Omega(\log n / \log \log n)}.$$

Proof compares digraph Ramsey numbers to ordered Ramsey numbers.

What about more than one color?

Definition

If $k \geq 1$, the k-color Ramsey number $\overrightarrow{r_k}(H)$ of a digraph H is the minimum N such that any k-edge-colored N-vertex tournament contains a monochromatic copy of H.

We showed: For $k = 1$, if H has n vertices and maximum degree Δ, then $\overrightarrow{r_1}(H) \leq n^{O_\Delta(\log n)}$, but $\overrightarrow{r_1}(H) \geq n^C$ is possible for any $C > 0$.

Theorem (Fox–H.–Wigderson 2021)

If H has n vertices and maximum degree Δ, then

$$\overrightarrow{r_k}(H) \leq n^{O_\Delta(\log^{O_k(1)} n)}.$$

For $k \geq 2$, there exists H of maximum degree 3 and

$$\overrightarrow{r_k}(H) \geq n^{\Omega(\log n / \log \log n)}.$$

Proof compares digraph Ramsey numbers to ordered Ramsey numbers.
Conlon–Fox–Lee–Sudakov and Balko–Cibulka–Král–Kynčl proved that random ordered matchings have large ordered Ramsey numbers.

Open questions

Let H have n vertices and maximum degree Δ.

Open questions

Let H have n vertices and maximum degree Δ.

Conjecture.

For any $C > 0$, there exist H with Δ independent of C for which $\vec{r}(H) \geq n^C$.

Open questions

Let H have n vertices and maximum degree Δ.

Conjecture.

For any $C > 0$, there exist H with Δ independent of C for which $\vec{r}(H) \geq n^C$.

Conjecture.

For a random H, $\vec{r}(H) = O_\Delta(n)$ almost surely.

Open questions

Let H have n vertices and maximum degree Δ.

Conjecture.

For any $C > 0$, there exist H with Δ independent of C for which $\vec{r}(H) \geq n^C$.

Conjecture.

For a random H, $\vec{r}(H) = O_\Delta(n)$ almost surely.

Question.

We know that $\vec{r}(H) = O(n)$ if H has bounded height or bounded bandwidth. Is there a single natural notion of "multiscale complexity" that captures both these parameters and implies $\vec{r}(H) = O(n)$?

Thank you!