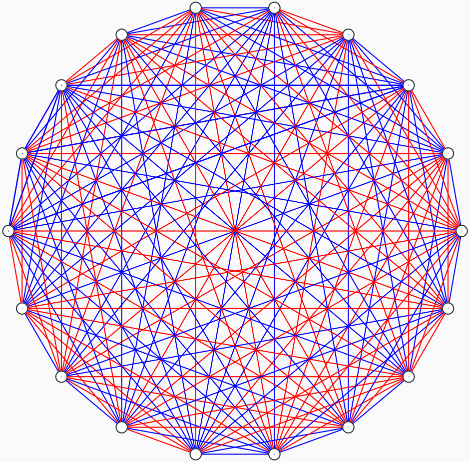
Set-coloring Ramsey Numbers via Codes

Xiaoyu He (Princeton University)

Joint work with D. Conlon, J. Fox, D. Mubayi, A. Suk, J. Verstraëte

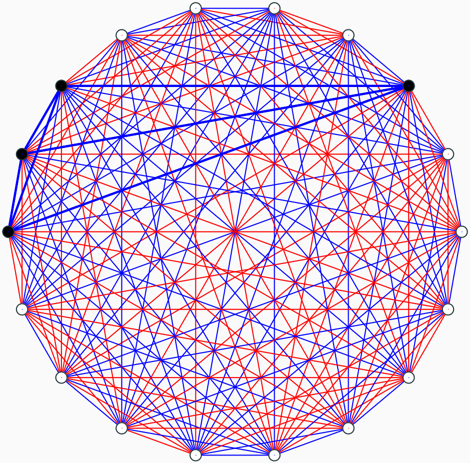
June 14, 2023

Introduction

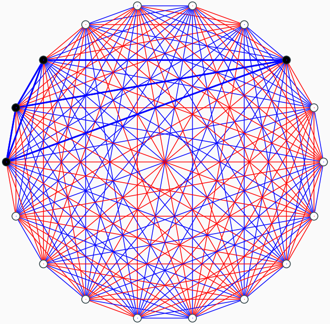


Complete disorder ...

Ramsey Theory



Complete disorder … is impossible.



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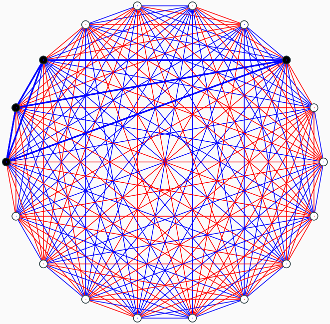

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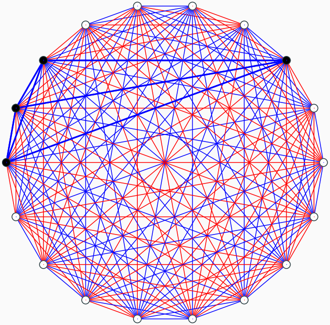
$R(4,4) = 18.$



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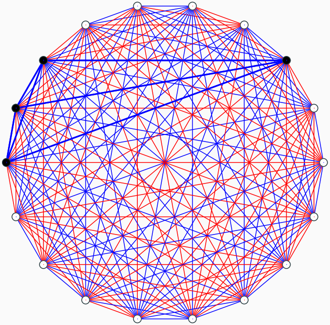
A Brief History of $R(s, t)$



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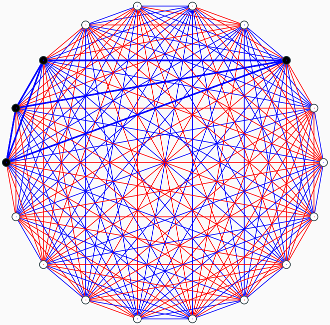

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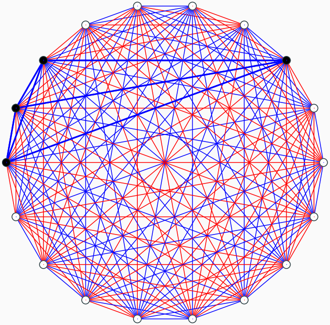

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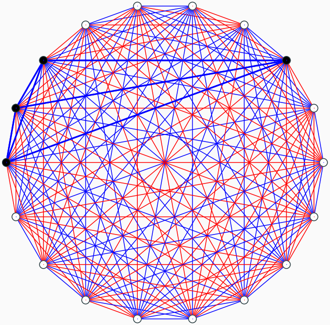

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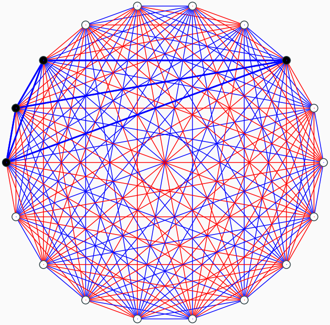

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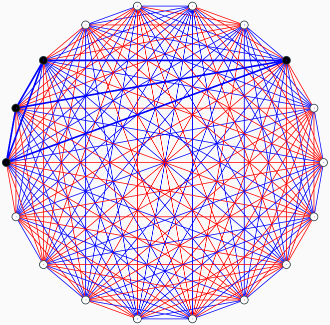

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Paradigm. How far can we push random lower bounds and the Erdős Szekeres upper bound?

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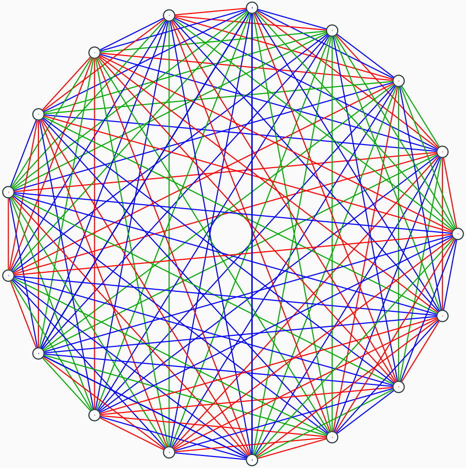

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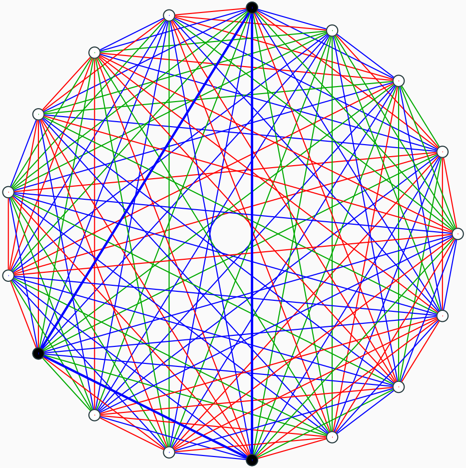
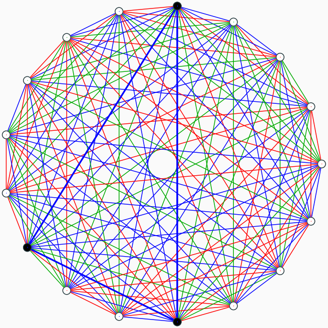


Complete disorder is still impossible, but more is possible than before.

Multicolor Ramsey numbers



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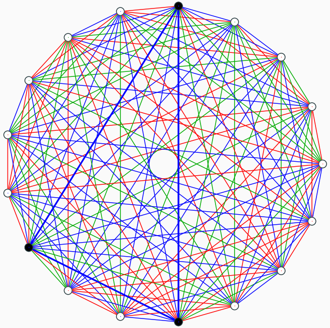

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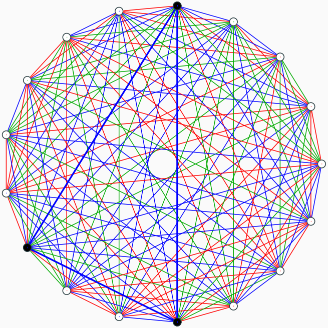
$R(3; 3) = 17$.



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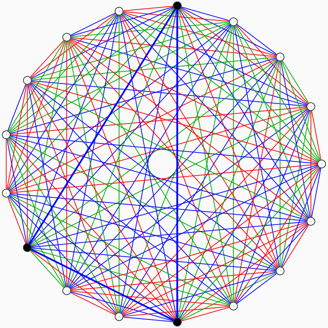

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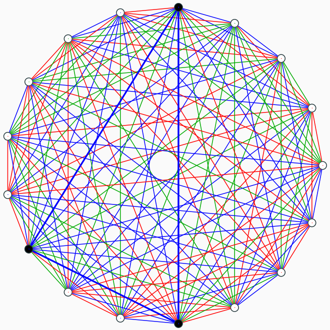

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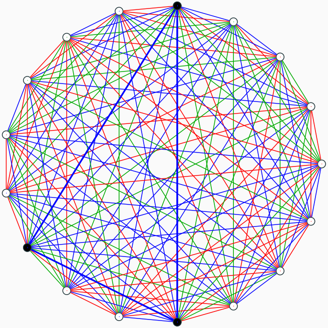

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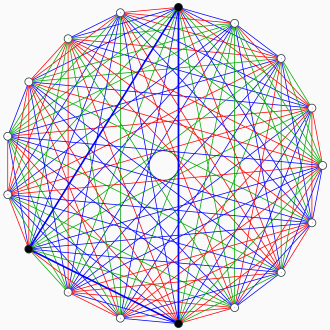

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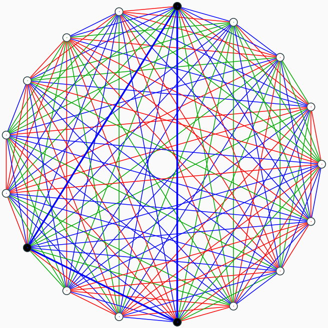

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Problem (Erdős $250). Is $R(t; r) = 2^{\Theta(rt)}$ or $2^{\Theta(rt \log r)}$?

Color-Avoiding Ramsey Numbers

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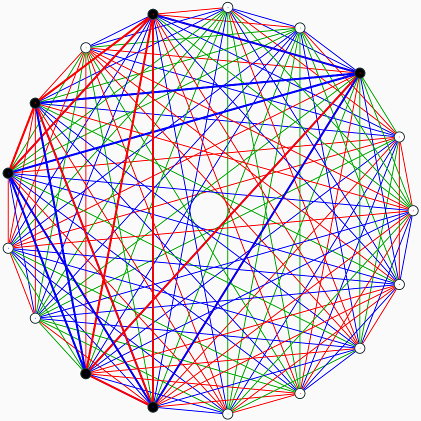

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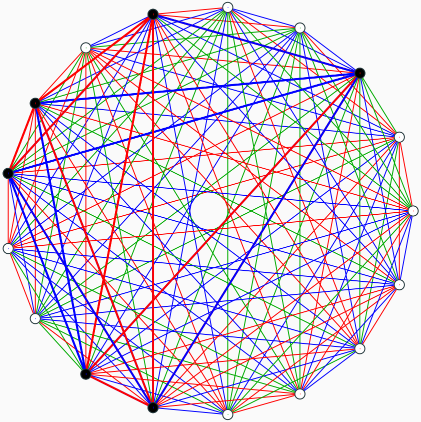

Erdős-Szemerédi '72

The smallest N for which any r-coloring of the edges of K_N contains a color-avoiding clique of size t satisfies $2^{\Omega(t/r)} \leq N \leq 2^{O(t \log r/r)}$

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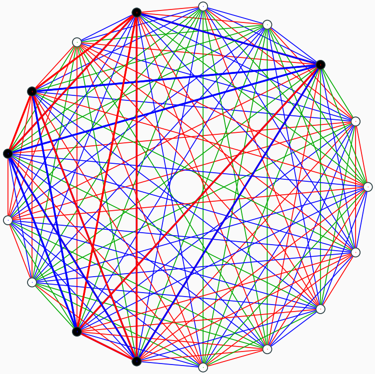
Set-coloring Ramsey numbers and various special cases have been studied by many previous authors. As far as I know the first appearance of the general case in the literature is due to Xu, Shao, Su, Li '10 in the guise of multi-graph Ramsey numbers.

Color-avoiding Ramsey number



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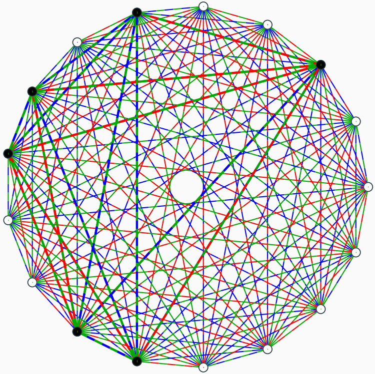

Color each edge on the right by the set of missing colors on the left.

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Define $R'(t; r, s)$ to be the smallest N such that if every edge of K_N is colored by one of r colors, then there is a clique of size t spanning at most s colors.

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There has been beautiful recent work on $R'(t; r, s)$ for hypergraphs by Dubroff, Girão, Hurley, and Yap.

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Theorem (Conlon, Fox, H., Mubayi, Suk, Verstraëte '22+)

If s/r is bounded away from 0 and 1, then $R(t;r,s) = 2^{\Theta(tr)}$.

Theorem (Erdős-Szemerédi '72)

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As long as $N > (r/s)^{tr} = 2^{O(tr)}$, this works.

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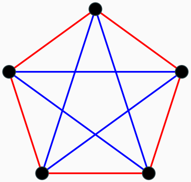

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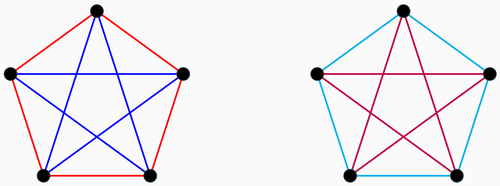

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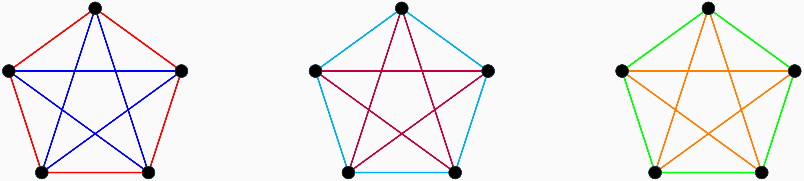

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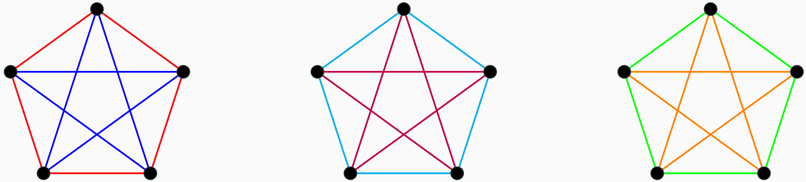

Pass to an induced subgraph on an error-correcting code of distance s to guarantee that each edge receives at least s colors.

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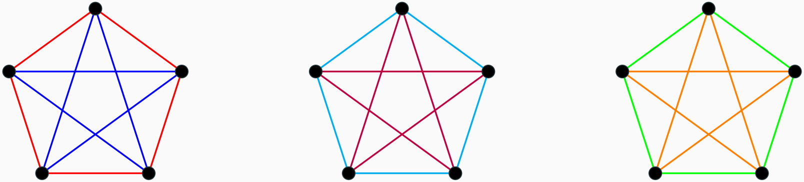
Let G be the Ramsey construction for $R(t;2,1)$ of size $2^{t/2}$ (random).

Take $r/2$ copies of G, where each new copies uses two new colors. Let H be the product of these copies, whose vertex set is $[2^{t/2}]^{r/2}$ and an edge (u,v) is colored by the color of **all** (u_i, v_i) where $u_i \neq v_i$.



Pass to an induced subgraph on an error-correcting code of distance s to guarantee that each edge receives at least s colors.

Such a code of size $2^{tr/4}/(r^s 2^{ts/2})$ exists, so we are done.

Followup Work

Theorem (Conlon, Fox, H., Mubayi, Suk, Verstraëte ’22+)

If s/r is bounded away from 0 and 1, then $R(t; r, s) = 2^{\Theta(tr)}$.

Followup Direction 1: what happens for s/r near 0 and 1?

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If $r > s \geq 1$ and $s = (1 - \varepsilon)r$, we have

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This settles the exponent up to logarithmic factors in all ranges. They replace our product+codes construction with random blowups+alterations (closer to the multicolor Ramsey constructions of Conlon-Ferber, Wigderson, Sawin).

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If t is fixed and s/r is less than and bounded away from $1 - \frac{1}{t-1}$, then $R(t; r, s)$ grows exponentially in r.

If t is fixed and s/r is greater than and bounded away from $1 - \frac{1}{t-1}$, then $R(t; r, s)$ is bounded.

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Theorem (Conlon, Fox, Pham, Zhao '23)

If t is fixed and s/r is close to $1 - \frac{1}{t-1}$, then $R(t; r, s)$ is tightly controlled by $A_{t-1}(r, s)$, the size of the largest error-correcting code with alphabet size $t-1$, length r, and distance s.

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Followup Direction 3: a connection hypergraph and grid Ramsey numbers

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Followup Direction 3: a connection hypergraph and grid Ramsey numbers

Let $S_n^{(3)}$ be the 3-uniform star, the hypergraph on $n+1$ vertices whose edges are all $\binom{n}{2}$ triples containing a given vertex. Let $K_n^{(3)}$ be the 3-uniform complete graph on n vertices.

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We have $2^{\Omega((\log n)^2)} \leq R(S_n^{(3)}, K_4^{(3)}) \leq 2^{n^{2/3+o(1)}}$, while $R(S_n^{(3)}, K_m^{(3)}) = 2^{\Theta(n)}$ for $m \geq 5$ fixed.

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The upper bound uses the Erdős-Szemerédi color-avoiding Ramsey number, while generalizations thereof use the set-coloring Ramsey number.

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The minimum N such that any red-blue coloring of the edges of $K_N \square K_N$ contains either a red rectangle (i.e. induced C_4) or a blue K_n satisfies

$$2^{\Omega((\log n)^2)} \leq N \leq 2^{n^{2/3+o(1)}}.$$

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The proofs use set-coloring Ramsey numbers and bear some similarity to the "pigeonhole one dimension at a time" arguments in the multi-dimensional Erdős-Szekeres problem.

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1. Close the logarithmic gap for $s = 1$:

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$$2^{\Omega(t/r)} \leq R(t; r, r-1) \leq 2^{O(t \log r / r)}.$$

3. Determine the minimum N such that any red-blue coloring of the edges of $K_N \square K_N$ contains either a red rectangle or a blue K_n.

$$2^{\Omega((\log n)^2)} \leq N \leq 2^{n^{2/3+o(1)}}.$$